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TWO CHANNEL PASSIVE SIGNAL DETECTION, (U)

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TWO CHANNEL PASSIVE SIGNAL DETECTION

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PREFACE

This report derives the optimum likelihood ratio receiver for detecting the presence of a Gaussian noise like signal masked by a background of Gaussian noise. Two channels of information are used to include the effect of a directive signal source. The mathematics of this problem is considered in some detail since it is applicable to the active sonar problem as well as the passive problem. The report is written for the reader who has not been exposed to the spatial processing problem but does have the necessary background in detection theory. For the more advanced reader wishing to pursue this topic, the report by Van Trees[1] and the paper by Edelblute,[2] *et al.* are highly recommended.

The author wishes to express his appreciation to Mr. D. Edelblute for his help with the matrix algebra and, particularly, to Dr. H.L. Van Trees for providing the necessary problem perspective.

Only limited distribution is contemplated.

1. Van Trees, H.L.; "A Unified Theory for Optimum Array Processing," A.D. Little, Inc., Report No. 416086C, August 1966.
2. Edelblute, D.J.; Fisk, J.M., and Kinnison, G.L., "Criteria for Optimum-Signal-Detection Theory for Arrays," JASA, *41, 1*, January 1967.

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INTRODUCTION

The optimum detection problem to be examined in this report is shown in figure 1. The waveforms $r_1(t)$ and $r_2(t)$ are observed at two spatially separated points for a finite time duration T, and we are asked to design the optimum receiver under the two hypotheses, "signal plus noise" (H_1) or "noise only" (H_0):

$$H_1: \begin{array}{l} r_1(t) = s(t) + n_1(t) \\ r_2(t) = s(t-\tau) + n_2(t) \end{array} \tag{1a}$$

$$H_0: \begin{array}{l} r_1(t) = n_1(t) \\ r_2(t) = n_2(t) \end{array} \tag{1b}$$

The target signal s(t) appearing on channel one is a member function from a stationary, zero-mean, Gaussian random process. Because of the spatial separation between channels, the target signal appearing on channel two is delayed by τ seconds and is $s(t-\tau)$. Physically, τ is related to the target bearing angle which is assumed to be constant during the observation interval. This implies τ is fixed.

The noise processes $n_1(t)$ and $n_2(t)$ appearing on channels one and two, respectively, are stationary, zero-mean, Gaussian random processes that are statistically dependent, but assumed to be statistically independent of the target signal processes. Thus, the problem formulated is the two channel Gauss signal in Gauss noise problem with dependent noise between channels.

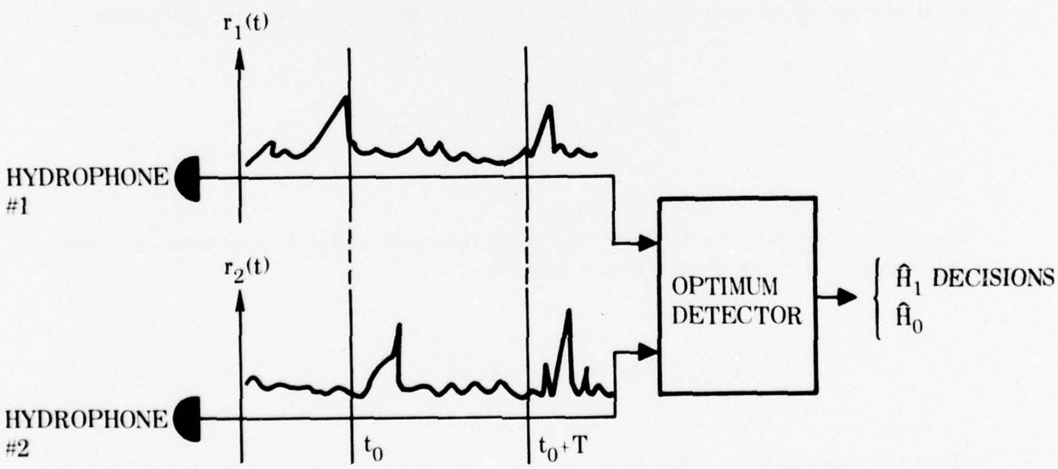

Figure 1. Two Channel Detection Problem.

VECTOR SPACE REPRESENTATION

The key to analyzing problems of this type is to find some way to convert all received functions of time into sets of numbers. Mathematically, this corresponds to finding a vector space which represents the random process in some sense. We will let

$$r(t) = \sum_{i=1}^{N} R_i \, \varphi_i(t) \tag{2a}$$

and require that the expected value of the mean-square difference between signal and its representation

$$E\left\{\int_I \left[r(t) - \sum_{i=1}^{N} R_i \, \varphi_i(t)\right]^2 dt\right\} \tag{2b}$$

converge to zero for $t \varepsilon I = [t_0, t_0 + T]$ as the number of terms $N \to \infty$. A consequence of requiring mean-square convergence is that the coefficients in the expansion will be obtained from

$$R_i = \int_I r(t) \, \varphi_i^*(t) \, dt \tag{3a}$$

where the set of basis functions $\{\varphi_i(t)\}$ have the property that they are orthonormal

$$\delta_{ij} = \int_I \varphi_i(t) \, \varphi_j^*(t) \, dt \tag{3b}$$

The choice of a particular $\{\varphi_i(t)\}$ to ensure convergence is due to Karhunen and Loève and requires solutions to the integral equation

$$\int_I K_r(t,u) \, \varphi_j^*(u) \, du = \lambda_j \varphi_j(t) \tag{4a}$$

where the covariance is defined as

$$K_r(t,u) = E\{[r(t) - m(t)][r(u) - m(u)]\} \tag{4b}$$

A particularly useful property of the Karhunen-Loève representation is that the choice of $\{\varphi_i(t)\}$ required by equation (4a) leads to a set of coefficients $\{R_i\}$ which are uncorrelated

$$E\{R_i R_j^*\} = \lambda_i \delta_{ij} \tag{5}$$

Since the processes we are dealing with are Gaussian, and furthermore, the mapping to obtain the coefficients as given by equation (3a) is linear, then the set $\{R_i\}$ must be jointly Gaussian random variables; hence, the uncorrelatedness of the coefficients implies statistical independence as well.

The foregoing discussion of our choice for a vector space is the correct way to deal with the problem. However, a number of practical problems arise which must be dealt with. The most difficult of these is the requirement for finding solutions to the integral equation given by equation (4a). That is, from knowledge of the covariance function, find the basis vectors, $\{\varphi_i(t)\}$. The additional requirement of letting $N \to \infty$ is prohibitive if we intend to calculate $\{R_i\}$ for use in the optimum detection system. Motivated by these two practical problems, we will choose the set of basis vectors to be

$$\{\varphi_n(t)\} \stackrel{\Delta}{=} \{e^{jn\omega_0 t}\} \tag{6}$$

where $\omega_0 = 2\pi/T$ and $n = 0, \pm 1, \pm 2, \ldots \pm N$. Thus, we have chosen a truncated Fourier representation. From equation (2a),

$$r(t) = \sum_{-N\omega_0}^{+N\omega_0} R(n\omega_0)\, e^{jn\omega_0 t} \tag{7a}$$

and from equation (3a)

$$R(n\omega_0) = \frac{1}{T} \int_{-T/2}^{T/2} r(t)\, e^{-jn\omega_0 t}\, dt \tag{7b}$$

which is the complex spectral component at frequency $n\omega_0$. Obviously, the choice of the Fourier basis will not, in general, provide solutions to the Karhunen-Loève integral equation, and, conversely, insure complete representation of the random process. Also, choosing the Fourier basis does not insure that the variables remain uncorrelated. With these factors in mind, let us take the integral product defined by equation (7b). We then have the set of complex Fourier coefficients forming the random vectors $\mathbf{R}_1$ and $\mathbf{R}_2$ for channels one and two, respectively;

$$\mathbf{R}_1 = \begin{bmatrix} R_1(\omega_u) \\ \vdots \\ R_1(\omega_l) \end{bmatrix} \qquad \mathbf{R}_2 = \begin{bmatrix} R_2(\omega_u) \\ \vdots \\ R_2(\omega_l) \end{bmatrix} \tag{8a}$$

with $\omega_u = N\omega_0$; $\omega_l = -N\omega_0$; and

$$R_i(\omega_k) = \frac{1}{T}\int_{-T/2}^{T/2} r_i(t)\, e^{-j\omega_k t}\, dt \qquad i = 1, 2 \tag{8b}$$

The desired property that the coefficients be uncorrelated

$$E\{R_i(\omega_m)\, R_j(\omega_n)\} = 0 \quad \begin{cases} m \neq n \\ i, j = 1, 2 \end{cases} \tag{8c}$$

for different frequencies is examined in the Appendix. It is shown there that uncorrelatedness holds if and only if (a) the process is periodic; that is, $K_r(t-u) = K_r(t-u+T)$ or (b) the observation interval becomes very long; $T \to \infty$.

LIKELIHOOD RATIO TEST: $\Lambda(\mathbf{R}_1, \mathbf{R}_2)$

Having established a way to convert functions of time into sets of numbers, our attention is turned to deriving the likelihood ratio. That the likelihood ratio is a sufficient scalar statistic for either the Bayes or the Neyman-Pearson criterion is well known.[3] Hence, we begin directly by defining the likelihood ratio as

$$\Lambda(\mathbf{R}_1, \mathbf{R}_2) \triangleq \frac{p\{\mathbf{R}_1, \mathbf{R}_2 \mid H_1\}}{p\{\mathbf{R}_1, \mathbf{R}_2 \mid H_0\}} \tag{9}$$

the ratio of the joint probability density functions conditioned on each hypothesis. The optimum detector is one which computes the likelihood ratio and compares it to a threshold. This rule is

$$\Lambda(\mathbf{R}_1, \mathbf{R}_2) \underset{H_0}{\overset{H_1}{\gtrless}} \eta \tag{10}$$

3. *Detection, Estimation, and Modulation Theory, Part 1;* H.L. Van Trees, John Wiley and Sons, Inc., New York, N.Y., 1967, pp. 24-34.

The choice of η is of course dependent upon the values and costs chosen for the Bayes criterion, or upon the allowable false alarm probability in the case of the Neyman-Pearson criterion. The main job remaining is to evaluate $\Lambda(\mathbf{R}_1, \mathbf{R}_2)$, the comparison to a threshold is trivial.

Simplification of $\Lambda(\mathbf{R}_1, \mathbf{R}_2)$

Rewriting the likelihood ratio using the definition for $\mathbf{R}_1$ and $\mathbf{R}_2$ gives

$$\Lambda(\mathbf{R}_1, \mathbf{R}_2) = \frac{p\{R_1(\omega_u)\ R_2(\omega_u) \ldots R_1(\omega_l)\ R_2(\omega_l) \mid H_1\}}{p\{R_1(\omega_u)\ R_2(\omega_u) \ldots R_1(\omega_l)\ R_2(\omega_l) \mid H_0\}} \tag{11a}$$

after grouping terms of the same frequency and reordering. By Bayes' rule this may be re-written as

$$\Lambda(\mathbf{R}_1, \mathbf{R}_2) = \frac{p\{R_1(\omega_u)R_2(\omega_u) \mid \ldots R_1(\omega_l)R_2(\omega_l) : H_1\}p\{\ldots R_1(\omega_l)R_2(\omega_l) \mid H_1\}}{p\{R_1(\omega_u)R_2(\omega_u) \mid \ldots R_1(\omega_l)R_2(\omega_l) : H_0\}p\{\ldots R_1(\omega_l)R_2(\omega_l) \mid H_0\}} \tag{11b}$$

Since the received process is gaussian under either hypothesis and, furthermore, that $\{R(\omega_i)\}$ are gaussian random variables because they were obtained by the linear operation of equation (8b), the probability distributions in equation (11b) are multivariate gaussian. But, we have already stated that (see equation 8c) spectral coefficients at different frequencies are uncorrelated. Thus, since the distributions are gaussian, this is equivalent to the coefficients being statistically independent, and equation (11b) becomes

$$\Lambda(\mathbf{R}_1 \mathbf{R}_2) = \frac{p\{R_1(\omega_u)R_2(\omega_u) \mid H_1\}p\{\ldots R_1(\omega_l)R_2(\omega_l) \mid H_1\}}{p\{R_1(\omega_u)R_2(\omega_u) \mid H_0\}p\{\ldots R_1(\omega_l)R_2(\omega_l) \mid H_0\}} \tag{12}$$

By the same reasoning, successive application of Bayes' rule to the remaining terms on the far right for each discrete frequency in the range will yield

$$\Lambda(\mathbf{R}_1, \mathbf{R}_2) = \prod_{k=1}^{u} \frac{p\{R_1(\omega_k)R_2(\omega_k) \mid H_1\}}{p\{R_1(\omega_k)R_2(\omega_k) \mid H_0\}} \tag{13}$$

$$= \prod_{k=1}^{u} \Lambda\Big(R_1(\omega_k), R_2(\omega_k)\Big)$$

The likelihood ratio reduces to the product of the likelihood ratio at each frequency, a useful simplification arising from uncorrelated coefficients for jointly gaussian random variables.

The crucial assumptions for equation (13) to hold are:

(a) The process under each hypothesis must be gaussian and stationary.

(b) The process must be Fourier expandable (implying either a periodic process or a long observation interval).

We state that (b) and the assumption of stationarity, may be relaxed if the eigenvectors in the Karhunen-Loève expansion are used (a solution for equation 4a has been found).

It remains now to determine the numerator and denominator terms in equation 13. However, before this can be done, we must first define the form of the bi-variate, complex Gaussian distribution since both $R_1(\omega_k)$ and $R_2(\omega_k)$ are complex.

Complex Bi-variate Gaussian Distribution

Suppressing the notation for the specific frequency ω_k, the pair of complex random variables, defined as

$$\begin{aligned} R_1 &= a_1 + i\,b_1 \\ R_2 &= a_2 + i\,b_2 \end{aligned} \tag{14a}$$

and having the assumed property that

$$E\{R_1\} = E\{R_2\} = 0 \tag{14b}$$

with a covariance matrix

$$\mathbf{K} = \begin{bmatrix} \sigma_{11}^{2} & \sigma_{12} \\ \sigma_{21} & \sigma_{22}^{2} \end{bmatrix} \tag{14c}$$

where

$$\begin{aligned} \sigma_{11}^2 &= E\{R_1 R_1^*\} & \sigma_{22}^2 &= E\{R_2 R_2^*\} \\ \sigma_{12} &= E\{R_1 R_2^*\} & \sigma_{21} &= E\{R_2 R_1^*\} \end{aligned} \tag{14d}$$

then the form for the bivariate distribution of $\mathbf{R} = \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}$ is

$$p\{R_1, R_2\} = \frac{1}{\pi^2|\mathbf{K}|} \exp - \left\{ (R_1, R_2)^* \, \mathbf{K}^{-1} \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} \right\} \tag{15a}$$

The inverse of the covariance matrix is

$$\mathbf{K}^{-1} = \frac{\begin{bmatrix} \sigma_{22}^2 & -\sigma_{12} \\ -\sigma_{12}^* & \sigma_{11}^2 \end{bmatrix}}{\sigma_{11}^2\sigma_{22}^2 - \sigma_{12}\sigma_{12}^*} \tag{15b}$$

when it is recognized that $\sigma_{12}^* = \sigma_{21}$. We note in passing that since $a_1(a_2)$ and $b_1(b_2)$ are two gaussian random variables which are statistically independent (orthogonal implies uncorrelated) then $R_1(R_2)$ is jointly gaussian and $|R_1|(|R_2|)$ is distributed Rayleigh and arg R_1 (arg R_2) is uniformly distributed. We are now prepared to evaluate each of the terms in equation (13).

Evaluation of $p\{R_1(\omega_k), R_2(\omega_k) \mid H_0\}$

Under hypothesis H_0, the received signals are;

$$\begin{aligned} r_1(t) &= n_1(t) \\ r_2(t) &= n_2(t) \end{aligned} \tag{16}$$

Finding the Fourier coefficients with equation (7b) at a particular frequency ω_k, we obtain (suppressing the notation for ω_k);

$$\begin{aligned} R_1 &= N_1 \\ R_2 &= N_2 \end{aligned} \tag{17}$$

The covariance matrix of the noise is;

$$\mathbf{K}_0 = \begin{bmatrix} \sigma_{11}^2 & \sigma_{12} \\ \sigma_{12}^* & \sigma_{22}^2 \end{bmatrix} \tag{18a}$$

where

$$\begin{aligned} \sigma_{11}^2 &= E\{N_1N_1^*\} \qquad \sigma_{22}^2 = E\{N_2N_2^*\} \\ \sigma_{12} &= E\{N_1N_2^*\} \qquad \sigma_{12}^* = E\{N_2N_1^*\} \end{aligned} \tag{18b}$$

From the form of the complex bi-variate gaussian distribution given by equation (15a), the conditional distribution under noise becomes

$$p\{R_1, R_2 \mid H_0\} = \frac{1}{\pi^2|\mathbf{K}_0|} \exp - \left\{ (R_1, R_2)^* \, \mathbf{K}_0^{-1} \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} \right\} \tag{19}$$

with $\mathbf{K}_0$ given by equation (18).

Evaluation of $p\{R_1(\omega_k), R_2(\omega_k) | H_1\}$

Under hypothesis H_1, the received signals are:

$$\begin{aligned} r_1(t) &= s(t) + n_1(t) \\ r_2(t) &= s(t-\tau) + n_2(t) \end{aligned} \tag{20}$$

Finding the Fourier coefficients with equation (7b) at a particular frequency ω_k, we obtain;

$$\begin{aligned} R_1(\omega_k) &= S(\omega_k) + N_1(\omega_k) \\ R_2(\omega_k) &= S(\omega_k)\, e^{-j\omega_k T} + N_2(\omega_k) \end{aligned} \tag{21a}$$

We wish to suppress the notation with ω_k. Consider that $\tau = \omega_k T$; we can rewrite (21a) as

$$\begin{aligned} R_1 &= S + N_1 \\ R_2 &= S\, e^{-j\tau} + N_2 \end{aligned} \tag{21b}$$

The covariance matrix for signal plus noise is;

$$\mathbf{K}_1 = \begin{bmatrix} E\{R_1 R_1^*\} & E\{R_1 R_2^*\} \\ E\{R_1 R_2^*\}^* & E\{R_2 R_2^*\} \end{bmatrix} \tag{22}$$

Evaluating each term in the matrix for R_1 and R_2 given by equation (21b) and knowledge that noise is independent of signal, we obtain;

$$\mathbf{K}_1 = \mathbf{K}_0 + E\{SS^*\} \begin{bmatrix} 1 & e^{+j\tau} \\ e^{-j\tau} & 1 \end{bmatrix} \tag{23}$$

The coefficient of the matrix is recognized as (see equation A-16 and A-19)

$$E\{S(\omega_k) S^*(\omega_k)\} = \Phi(\omega_k) \tag{24}$$

the signal power spectrum evaluated at the frequency ω_k.

If we define a column vector to be

$$\mathbf{V} = \begin{bmatrix} 1 \\ e^{-j\tau} \end{bmatrix} \tag{25}$$

then equation 23 can be written compactly as

$$\mathbf{K}_1 = \mathbf{K}_0 + \Phi\, \mathbf{V}\mathbf{V}^\dagger \tag{26}$$

with $(\ )^\dagger$ representing $(\ )^{*T}$, the transpose of the complex conjugate.

From the form of the complex bi-variate gaussian distribution given by equation (15a), the conditional distribution under signal plus noise becomes

$$p\{R_1, R_2 \mid H_1\} = \frac{1}{\pi^2 |\mathbf{K}_1|} \exp - \left\{ (R_1, R_2)^* \, \mathbf{K}_1^{-1} \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} \right\} \tag{27}$$

with $\mathbf{K}_1$ given by equation 26.

Algebraic Reduction of $\Lambda(\mathbf{R}_1, \mathbf{R}_2)$

Returning to the likelihood ratio of equation 13, and substituting for both of the conditional density functions given by equation 19 and equation 27, we have

$$\Lambda(\mathbf{R}_1, \mathbf{R}_2) = \prod_{k=1}^{u} \frac{|\mathbf{K}_0|}{|\mathbf{K}_1|} \exp - \left\{ (R_1, R_2)^* \left[\mathbf{K}_1^{-1} - \mathbf{K}_0^{-1} \right] \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} \right\} \tag{28}$$

As is usually the case, we note that comparing $\Lambda(\mathbf{R}_1, \mathbf{R}_2)$ to a threshold for our optimum decisions is equivalent to comparing $\ln \Lambda(\mathbf{R}_1, \mathbf{R}_2)$ to some other threshold since $\exp - \{\ \}$ is a monotonic function of its argument. Thus, we may rewrite our decision rule of equation 10 as

$$\Lambda'(\mathbf{R}_1, \mathbf{R}_2) = \sum_{k=1}^{u} (R_1, R_2)^* \left[\mathbf{K}_0^{-1} - \mathbf{K}_1^{-1} \right] \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} \underset{H_0}{\overset{H_1}{\gtrless}} \gamma \tag{29a}$$

with the new threshold

$$\gamma = \ln \eta - \sum_{k=1}^{u} \ln \left\{ \frac{|\mathbf{K}_0|}{|\mathbf{K}_1|} \right\} \tag{29b}$$

The term $[\mathbf{K}_0^{-1} - \mathbf{K}_1^{-1}]$ can be reduced algebraically as follows. Substituting for $\mathbf{K}_1$ from equation 26 gives

$$\begin{aligned} [\mathbf{K}_0^{-1} - \mathbf{K}_1^{-1}] &= \mathbf{K}_0^{-1} - [\mathbf{K}_0 + \Phi\, \mathbf{V}\mathbf{V}^\dagger]^{-1} \\ &= \mathbf{K}_0^{-1} - [\mathbf{K}_0(\mathbf{I} + \Phi\, \mathbf{K}_0^{-1}\, \mathbf{V}\mathbf{V}^\dagger)]^{-1} \\ &= \mathbf{K}_0^{-1} - [\mathbf{I} + \Phi\, \mathbf{K}_0^{-1}\, \mathbf{V}\mathbf{V}^\dagger]^{-1}\, \mathbf{K}_0^{-1} \end{aligned} \tag{30}$$

We claim that the following identity is true

$$[\mathbf{I} + \Phi\, \mathbf{K}_0^{-1}\, \mathbf{V}\mathbf{V}^\dagger]^{-1} = \mathbf{I} - \frac{\Phi}{1 + \Phi\, \mathbf{V}^\dagger\, \mathbf{K}_0^{-1}\, \mathbf{V}}\, \mathbf{K}_0^{-1}\, \mathbf{V}\mathbf{V}^\dagger \tag{31}$$

To prove that this is so, we require that the righthand side of equation 31, when multiplied by the inverse of the lefthand side, yield the identity matrix. This is demonstrated for a somewhat more general case by Edelblute, *et al.*[4] Substituting into equation 30 gives;

$$[\mathbf{K}_0^{-1} - \mathbf{K}_1^{-1}] = \mathbf{K}_0^{-1} - \left\{ \mathbf{I} - \frac{\Phi}{1 + \Phi\, \mathbf{V}^\dagger\, \mathbf{K}_0^{-1}\, \mathbf{V}}\, \mathbf{K}_0^{-1}\, \mathbf{V}\mathbf{V}^\dagger \right\} \mathbf{K}_0^{-1}$$

$$= \frac{\Phi}{1 + \Phi\, \mathbf{V}^\dagger\, (\mathbf{K}_0^{-1}\, \mathbf{V})}\, (\mathbf{K}_0^{-1}\, \mathbf{V})(\mathbf{V}^\dagger\, \mathbf{K}_0^{-1})$$

$$= \frac{\Phi}{1 + \Phi\, \mathbf{V}^\dagger (\mathbf{K}_0^{-1}\, \mathbf{V})}\, (\mathbf{K}_0^{-1}\, \mathbf{V})(\mathbf{K}_0^{-1}\, \mathbf{V})^\dagger \tag{32}$$

the last step resulting from knowledge that $\mathbf{K}_0^{-1}$ is hermetian; $\mathbf{K}_0^{-1} = (\mathbf{K}_0^{-1})^\dagger$ by inspection of equation (15b). Substituting equation 32 into equation (29a) we have a somewhat simplified form for the log likelihood ratio;

$$\Lambda'(\mathbf{R}_1, \mathbf{R}_2) = \sum_{k=1}^{u} (R_1, R_2)^* \left[\frac{\Phi}{1 + \Phi\, \mathbf{V}^\dagger (\mathbf{K}_0^{-1}\, \mathbf{V})}\, (\mathbf{K}_0^{-1}\, \mathbf{V})(\mathbf{K}_0^{-1}\, \mathbf{V})^\dagger \right] \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} \tag{33}$$

Recall that every term inside the summation is a function of ω_k. By defining the vectors

$$\mathbf{R}(\omega_k) \triangleq \begin{pmatrix} R_1(\omega_k) \\ R_2(\omega_k) \end{pmatrix} \tag{34a}$$

$$\mathbf{Z}(\omega_k) \triangleq \begin{pmatrix} Z_1(\omega_k) \\ Z_2(\omega_k) \end{pmatrix} = \mathbf{K}_0^{-1}(\omega_k)\, \mathbf{V}(\omega_k) \tag{34b}$$

and the coefficient

$$\lambda(\omega_k) = \frac{\Phi(\omega_k)}{1 + \Phi(\omega_k)\, \mathbf{V}^\dagger(\omega_k)\, [\mathbf{K}_0^{-1}(\omega_k)\, \mathbf{V}(\omega_k)]} \tag{34c}$$

---

4. *Ibid;* p. 203.

The log-likelihood ratio can be written compactly as

$$\Lambda'(\mathbf{R}_1, \mathbf{R}_2) = \sum_{k=1}^{u} [\lambda^{1/2}(\omega_k)\, \mathbf{Z}^{\dagger}(\omega_k)\, \mathbf{R}(\omega_k)]^{\dagger}\, [\lambda^{1/2}(\omega_k)\, \mathbf{Z}^{\dagger}(\omega_k)\, \mathbf{R}(\omega_k)] \tag{35}$$

RECEIVER STRUCTURE

We may now determine the form of the optimum receiver which implements equation 35. As shown in figure 2, the two channels are Fourier transformed, resulting in the set of complex Fourier coefficients. These are then used to compute the log-likelihood ratio test statistic. This scalar is then compared to a constant which is a function of the signal power, the noise power, and *a priori* probabilities and costs. If the number is greater than this threshold, we make the decision that signal plus noise is in our observation interval; otherwise just noise. Thus, the receiver of figure 2 could be implemented on a digital computer and the hypothesis test run for each observation interval. There is, however, an equivalent receiver which could be implemented using realizable time-invariant filters. We shall state the result and then show it is true.

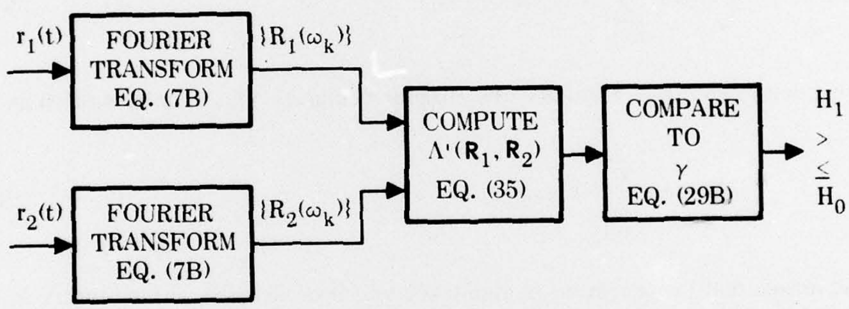

Figure 2. Computer Implementation of Two Channel Log-Likelihood Ratio Receiver.

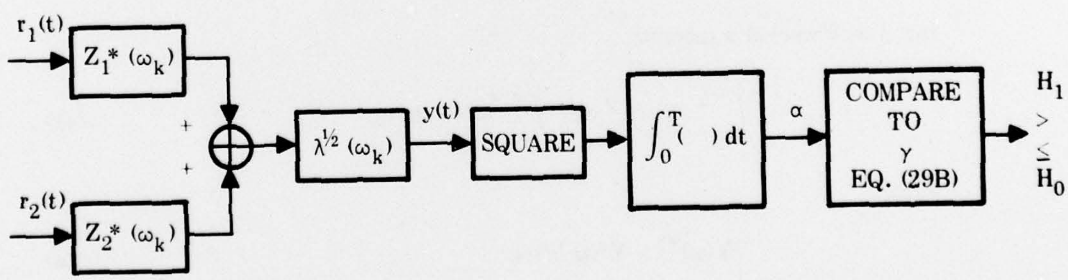

Figure 3. Frequency Transfer Function Implementation of Two Channel Log-Likelihood Ratio Receiver.

We claim that figure 3 is the frequency transfer representation for figure 2. Demonstration that this is true centers on showing that the test statistic α is identical to $\Lambda'(\mathbf{R}_1, \mathbf{R}_2)$. By inspection of figure 3, the output waveform $y(t)$ has a discrete frequency function $Y(\omega_k)$ given as

$$Y(\omega_k) = \lambda^{1/2}(\omega_k)[Z_1^*(\omega_k) R_1(\omega_k) + Z_2^*(\omega_k) R_2(\omega_k)] \tag{36}$$

In vector notation this is

$$Y(\omega_k) = \lambda^{1/2}(\omega_k)\, \mathbf{Z}^\dagger(\omega_k)\, \mathbf{R}(\omega_k) \tag{37a}$$

where

$$\begin{aligned} \mathbf{Z}^\dagger(\omega_k) &= [Z_1^*(\omega_k), Z_2^*(\omega_k)] \\ \mathbf{R}(\omega_k) &= \begin{bmatrix} R_1(\omega_k) \\ R_2(\omega_k) \end{bmatrix} \end{aligned} \tag{37b}$$

The continuous frequency function $Y(\omega)$ can be written as

$$Y(\omega) = \sum_{k=1}^{u} Y(\omega_k)\, \delta(\omega - \omega_k) \tag{38}$$

using delta functions. Similarly, the complex conjugate $Y^*(\omega)$ can be written as

$$Y^*(\omega) = \sum_{k=1}^{u} Y^\dagger(\omega_k)\, \delta(\omega - \omega_k) \tag{39}$$

The reason that the transpose is also taken will become obvious; note that $Y(\omega_k)$ is not a vector and, hence, the transpose has no effect.

Now, the test statistic α is, from figure 3,

$$\alpha = \int_0^T y^2(t)\, dt \tag{40}$$

But from Parseval's theorem,

$$\int_{\omega_l}^{\omega_u} |Y(\omega)|^2\, df = \int_0^T y^2(t)\, dt \tag{41}$$

Since

$$|Y(\omega)|^2 = Y(\omega)\, Y^*(\omega) \tag{42a}$$

we have, when use is made of equation 38 and equation 39,

$$|Y(\omega)|^2 = \sum_{k=1}^{u} Y(\omega_k)\, Y^{\dagger}(\omega_k)\, \delta(\omega - \omega_k) \tag{42b}$$

Using the definition of $Y(\omega)$ from equation (37a) and integrating over the range of ω as indicated by equation 41 yields

$$\alpha = \sum_{k=1}^{u} [\lambda^{1/2}(\omega_k)\, \mathbf{Z}^{\dagger}(\omega_k)\, \mathbf{R}(\omega_k)]^{\dagger}\, [\lambda^{1/2}(\omega_k)\, \mathbf{Z}^{\dagger}(\omega_k)\, \mathbf{R}(\omega_k)] \tag{43}$$

which is identical to $\Lambda'(\mathbf{R}_1, \mathbf{R}_2)$. Thus, the two receivers are identical.

SUMMARY

This report considers a very special problem, the two channel detection of a Gaussian noise-like signal masked by a background of Gaussian noise. The signal process was related between channels by a fixed time delay and the noise was correlated between channels. Implicit in the receiver structure of figures 2 and 3 is knowledge of this delay, τ. This is because $Z_1(\omega_k)$ and $Z_2(\omega_k)$ are functions of $\mathbf{V}(\omega_k)$ which in turn is related to τ; [see equation 25 and equation (34b)]. Thus, if τ is unknown and non-random, we have to first optimally estimate τ(equivalent to bearing angle) and then use this estimate in our optimum detector. It can be shown that if we choose a maximum *a posteriori* estimate, then the cascade of this estimate followed by the detector is jointly optimum. This problem will be considered at a future date along with some other more realistic models of the signal process.

The choice of a Fourier basis for our vector space was motivated by the knowledge that the signal spectral components on channel two would be just a phase shifted version of those appearing on channel one, and, thereby, greatly simplifying the mathematics of the problem.

The issue of how our optimum receiver performs has been left unresolved for a number of reasons. First, upon inspection of our test statistic given by equation 43, we see that it is of the form

$$\alpha = \sum^{M} a_i\, R_i^2 \tag{44}$$

It is immediately apparent that α is not distributed gaussian since we are dealing with the squares of complex gaussian random variables. Thus, determination of how α is distributed under each hypothesis in order to calculate the ROC curves is a difficult analytical task. Furthermore, it is not clear that this would give us the correct result since we have truncated the series to M terms in our vector space representation. It is speculated that a more reasonable approach is to bound the error probability for our receiver and determine how these bounds behave as a function of the delay τ (incorrect estimates of bearing) and of the correlation between channels (diagonal versus non-diagonal noise covariance matrices).

APPENDIX: THE CORRELATION BETWEEN FOURIER COEFFICIENTS

Let the signal on channel i be a sample function, $r_i(\theta,t)$ $[\theta \epsilon \Theta]$ from a wide-sense stationary, zero-mean random process. Now, suppose[1] the signal is expanded on a finite interval $-T/2 \leq t \leq T/2$ in a Fourier series;

$$r_i(\theta,t) = \sum_{n=-\infty}^{\infty} X_i(n\omega_0,\theta)e^{jn\omega_0 t} \tag{A-1}$$

where

$$\omega_0 = \frac{2\pi}{T} \tag{A-2}$$

$$X_i(n\omega_0,\theta) = \frac{1}{T}\int_{-T/2}^{T/2} r_i(\theta,t)e^{-jn\omega_0 t}dt \tag{A-3}$$

Different sample functions $r_i(\theta,t)$ yield in general different values $X_i(n\omega_0,\theta)$ since the process is random. Thus, the Fourier coefficient $X_i(n\omega_0)$ is a random variable and $X_i(n\omega_0,\theta)$ is the value the random variable takes on for all $\theta \epsilon \Theta$. With this distinction, and knowledge that expectation $E\{X_i(n\omega_0)\}$ is with respect to all sample points $\theta \epsilon \Theta$, the specific notation of θ will be dropped.

For the calculations shown in simplifying the likelihood ratio it was extremely convenient for the Fourier coefficients defined by Eq. A-3 to have the property that they be uncorrelated

$$E\{X_i(n\omega_0)X_j^*(m\omega_0)\} = 0 \qquad n \neq m \tag{A-4}$$

for all i, j and $n \neq m$. To see if this is so for a zero-mean, stationary process, some identities must first be developed. From Eq. A-3,

$$X_i(n\omega_0) = \frac{1}{T}\int_{-T/2}^{T/2} r_i(\xi)e^{-jn\omega_0\xi}d\xi \tag{A-5}$$

Taking the complex conjugate of both sides,

$$X_i^*(n\omega_0) = \frac{1}{T}\int_{-T/2}^{T/2} r_i^*(\xi)e^{+jn\omega_0\xi}\,d\xi \tag{A-6}$$

multiplying by $r_j(t)$ and taking the expectation

$$E\{r_j(t)\,X_i^*(n\omega_0)\} = \frac{1}{T}\int_{-T/2}^{T/2} E\{r_j(t)r_i^*(\xi)\}\,e^{+jn\omega_0\xi}\,d\xi \tag{A-7}$$

[1]We assume that the process can be represented in the mean-square sense by the Fourier series. As will be shown, this requires that the process be periodic or that the record length be infinite.

reduces to

$$E\{r_j(t)\,X_i^*(n\omega_0)\} = \frac{1}{T}\int_{-T/2}^{T/2} R_{ij}(t,\xi)e^{+jn\omega_0\xi}\,d\xi \qquad \text{(A-8)}$$

Since the process is stationary, $R_{ij}(t,\xi)$ is a function only of $t-\xi=\tau$. Making this substitution, A-8 becomes

$$E\{r_j(t)\,X_i^*(n\omega_0)\} = \frac{1}{T}\int_{t+T/2}^{t-T/2} R_{ij}(\tau)e^{+jn\omega_0(t-\tau)}\,d(-\tau) \qquad \text{(A-9a)}$$

$$= e^{+jn\omega_0 t}\left[\frac{1}{T}\int_{t+T/2}^{t-T/2} R_{ij}(\tau)e^{+jn\omega_0(-\tau)}\,d(-\tau)\right] \qquad \text{(A-9b)}$$

$$= e^{+jn\omega_0 t}\left[\frac{1}{T}\int_{t-T/2}^{t+T/2} R_{ij}(\tau)e^{-jn\omega_0\tau}\,d\tau\right]. \qquad \text{(A-9c)}$$

The bracket term is recognized as the Fourier transform of the cross-correlation function, which would be identical to the cross-spectral density if it were not for the variable t in the limits of integration. To eliminate the variable t, we can choose $R_{ij}(\tau+T)=R_{ij}(\tau)$; let the process be periodic. In this instance, the integral over a period T taken at any time t will be the same and A-9c reduces to

$$E\{r_j(t)\,X_i^*(n\omega_0)\} = e^{+jn\omega_0 t}\,\Phi_{ij}(n\omega_0) \qquad \text{(A-10a)}$$

where

$$\Phi_{ij}(n\omega_0) \triangleq \frac{1}{T}\int_{-T/2}^{T/2} R_{ij}(\tau)e^{-jn\omega_0\tau}\,d\tau\ . \qquad \text{(A-10b)}$$

To avoid assuming a periodic process, simplification of Eq. A-9c can be handled by letting T become large. Multiplying both sides of Eq. A-9c by T

$$TE\{r_j(t)\,X_i^*(n\omega_0)\} = -e^{+jn\omega_0 t}\int_{t+T/2}^{t-T/2} R_{ij}(\tau)e^{-jn\omega_0\tau}\,d\tau \qquad \text{(A-11)}$$

Papoulis[2] shows that by considering a large T an error $\varepsilon(t,T)$ is committed which tends to zero as $T\rightarrow\infty$.

$$TE\{r_j(t)\,X_i^*(n\omega_0)\} = e^{jn\omega_0 t}\,\Phi_{ij}(n\omega_0) + \varepsilon(t,T) \qquad \text{(A-12a)}$$

where

$$\Phi_{ij}(n\omega) = \int_{-T/2}^{T/2} R_{ij}(\tau)e^{-jn\omega_0\tau}\,d\tau \qquad \text{(A-12b)}$$

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[2] *Probability, Random Variables and Stochastic Processes*, A. Papoulis, McGraw Hill, New York, 1965, pp.454-456.

With the results of Eqs. A-10, and A-12, it will be possible to examine Eq. A-4. From Eq. A-3,

$$X_i(m\omega_0) = \frac{1}{T}\int_{-T/2}^{T/2} r_i(\xi)e^{-jm\omega_0\xi}d\xi . \qquad \text{(A-13)}$$

Taking the complex conjugate, multiplying by $X_j(n\omega_0)$, and taking the expectation

$$E\{X_j(n\omega_0)X_i^*(m\omega_0)\} = \frac{1}{T}\int_{-T/2}^{T/2} E\{X_j(n\omega_0)r_i^*(\xi)\}e^{+jm\omega_0\xi}d\xi \qquad \text{(A-14)}$$

The term $E\{X_j(n\omega_0)r_i^*(\xi)\}$ is identical to the complex conjugate of Eq. A-10a with t replaced by ξ, when $R_{ij}(\tau)$ is periodic. Substituting,

$$E\{X_j(n\omega_0)X_i^*(m\omega_0)\} = \frac{1}{T}\int_{-T/2}^{T/2} e^{-jn\omega_0\xi}\Phi_{ij}^*(n\omega_0)e^{+jm\omega_0\xi}d\xi \qquad \text{(A-15a)}$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} \Phi_{ij}^*(n\omega_0)e^{+j(m-n)\omega_0\xi}d\xi \qquad \text{(A-15b)}$$

$$= \Phi_{ij}^*(n\omega_0)\frac{1}{T}\int_{-T/2}^{T/2} e^{+j(m-n)\omega_0\xi}d\xi \qquad \text{(A-15c)}$$

The integral is identical to zero for $m \neq n$ and Eq. A-15c can be written as

$$E\{X_i(m\omega_0)X_j^*(n\omega_0)\} = \begin{cases}\Phi_{ij}(n\omega_0) & m = n\\ 0 & m \neq n\end{cases} \qquad \text{(A-16)}$$

after taking the complex conjugate. Thus, for a periodic process, the Fourier coefficients are uncorrelated.

When the process is not periodic, Eq. A-14 must be considered in the limit as $T \to \infty$. By taking the complex conjugate of Eq. A-12a,

$$E\{X_j(n\omega_0)r_i^*(t)\} = \frac{e^{-jn\omega_0 t}\Phi_{ij}^*(n\omega_0)}{T} + \frac{\varepsilon^*(t,T)}{T} . \qquad \text{(A-17)}$$

Substituting into Eq. A-14 after a change of variable from t to ξ gives

$$TE\{X_j(n\omega_0)X_i^*(n\omega_0)\} = \frac{\Phi_{ij}^*(n\omega_0)}{T}\int_{-T/2}^{T/2} e^{j(m-n)\omega_0\xi}d\xi$$

$$+ \frac{1}{T}\int_{-T/2}^{T/2} \varepsilon^*(\xi,T)e^{+jm\omega_0\xi}d\xi \qquad \text{(A-18)}$$

from which we conclude[3]

$$\lim_{T \to \infty} T E\{X_i(m\omega_0) X_j^*(n\omega_0)\} = \begin{cases} \Phi_{ij}(n\omega_0) & m = n \\ 0 & m \neq n \end{cases} \quad . \tag{A-19}$$

Thus, we may conclude that the Fourier coefficients are uncorrelated (Eq. A-4 holds) only when (1) the process is periodic or (2) when the observation interval becomes infinite.

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[3]ibid, pp.456